Journal of Mathematical Chemistry 25 (1999) 105–110

A new transformation for the Lotka–Volterra problem

C.M. Evans and G.L. Findley

Department of Chemistry, Northeast Louisiana University, Monroe, LA 71209, USA

Received 12 August 1998; revised 16 November 1998

The Lotka–Volterra dynamical system ($\dot{x}_1 = ax_1 - bx_1x_2,\ \dot{x}_2 = -cx_2 + bx_1x_2$) is reduced to a single second-order autonomous ordinary differential equation by means of a new variable transformation. Formal analytic solutions are presented for this latter differential equation.

The Lotka–Volterra (LV) problem consists of the following pair of first-order autonomous ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1x_2, \\ \dot{x}_2 &= -cx_2 + bx_1x_2, \end{aligned} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are real functions of time, $\dot{x}_i := \mathrm{d}x_i/\mathrm{d}t$, and a, b, c are positive real constants. This system was originally introduced by Lotka [8] in 1920 as a model of undamped oscillations in autocatalytic chemical reactions, and was later applied by Volterra [17] to treat predator–prey interactions in ecology. Other applications have followed in the intervening years in physics [13], chemistry [11], population biology [15] and epidemiology [14]. Indeed, the LV dynamical system is today a standard textbook example in the theory of nonlinear ordinary differential equations [10,16].

Since the original publication by Lotka [8], it has been known that equations (1) possess a dynamical invariant, namely,

$$\Lambda = bx_1 + bx_2 - c\ln x_1 - a\ln x_2. \tag{2}$$

By means of a logarithmic transformation, Kerner [5] showed that Λ serves to reduce equations (1) to a Hamiltonian system. This has recently sparked a resurgence of interest in the LV problem (including a rediscovery of some previously known results [6,12]), particularly with regard to dynamical invariants of generalizations of equations (1) [7,13].

Although the existence of a dynamical invariant for equations (1) implies that this system is solvable, very little is known about the analytic form of these solutions, with the exception of a Lie series analysis [3,4]. The purpose of the present note is to present a new transformation that reduces the LV system to a single second-order autonomous ordinary differential equation, and to reduce the solution of this equation to an integral quadrature. Thus, formal analytic solutions to equations (1) will be presented.

We begin by defining new coordinates $z_1(t)$ and $z_2(t)$ as follows:

$$z_1 = \Lambda^{-1/2}(bx_1 + bx_2)^{1/2}, \qquad z_2 = \Lambda^{-1/2}(-c\ln x_1 - a\ln x_2)^{1/2}, \tag{3}$$

where the positive square root is implied in each case. From equations (1)–(3) we find

$$x_1 = \frac{\Lambda}{b(a+c)}\left(cz_1^2 - 2z_2\dot{z}_2\right), \qquad x_2 = \frac{\Lambda}{b(a+c)}\left(az_1^2 + 2z_2\dot{z}_2\right), \tag{4}$$

and

$$z_1^2 + z_2^2 = 1. \tag{5}$$

Equation (5) permits the definition of an angle ϕ such that

$$z_1 = \sin\phi, \qquad z_2 = \cos\phi. \tag{6}$$

From equations (2), (4) and (6), then, we find

$$\ddot{\phi} + \left[\cot\phi - \tan\phi - \frac{2\Lambda}{a+c}\cos\phi\sin\phi\right]\dot{\phi}^2 + (c-a)\left(1 - \frac{\Lambda}{a+c}\sin^2\phi\right)\dot{\phi}$$
$$- \frac{1}{2}ac\left(1 - \frac{\Lambda}{a+c}\right)\tan\phi - \frac{1}{2}\frac{ac\Lambda}{a+c}\sin\phi\cos\phi = 0. \tag{7}$$

Making the substitution

$$w = \frac{\Lambda}{2(a+c)}(1 - \cos 2\phi), \tag{8}$$

equation (7) becomes

$$\ddot{w} - \dot{w}^2 - (c-a)(w-1)\dot{w} + acw(w-1) = 0. \tag{9}$$

Moreover, from equations (4), (6) and (8) it follows that

$$x_1 = \frac{1}{b}(cw + \dot{w}), \qquad x_2 = \frac{1}{b}(aw - \dot{w}). \tag{10}$$

Equation (9) is fully equivalent to the LV dynamical system (i.e., equations (1)). A fourth-order Runge–Kutta [1] integration of equation (9) is shown in figure 1 for a typical trajectory, and the usual phase-plane plot of equations (10) for this same trajectory is given in figure 2. (These results are identical with those we have obtained from a direct numerical integration of equations (1), the plots of which are not shown for the sake of brevity.)

If we now set $c = \alpha a$, equation (2) can be rearranged to yield

$$bx_1x_2 = -\frac{k^2}{b}x_1^{1-\alpha}\mathrm{e}^{b(x_1+x_2)/a}, \quad k^2 = -b^2\mathrm{e}^{-\Lambda/a}, \tag{11}$$

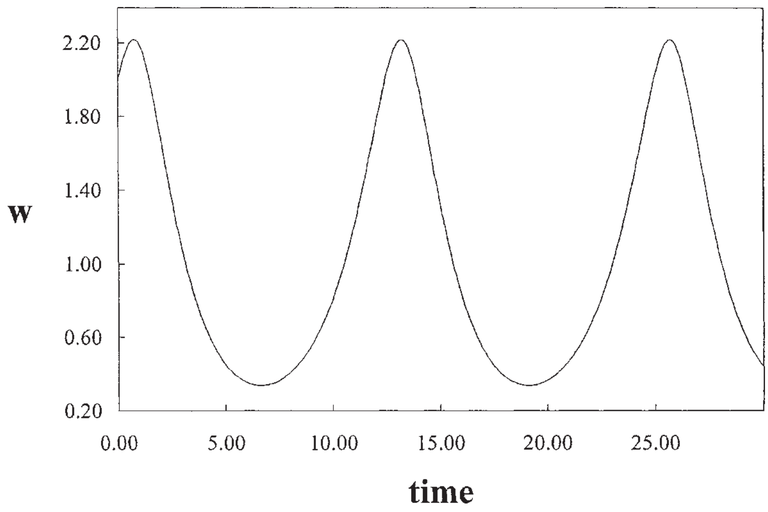

Figure 1. Fourth-order Runge–Kutta solution to equation (9) of the text for $a = 0.50,\ b = 1.30,\ c = 0.67$. Initial data for this trajectory are $w(t = 0) = 2.00,\ \dot{w}(t = 0) = 0.50$. The invariant (equation (2) of text) is $\Lambda = 2.5850$.

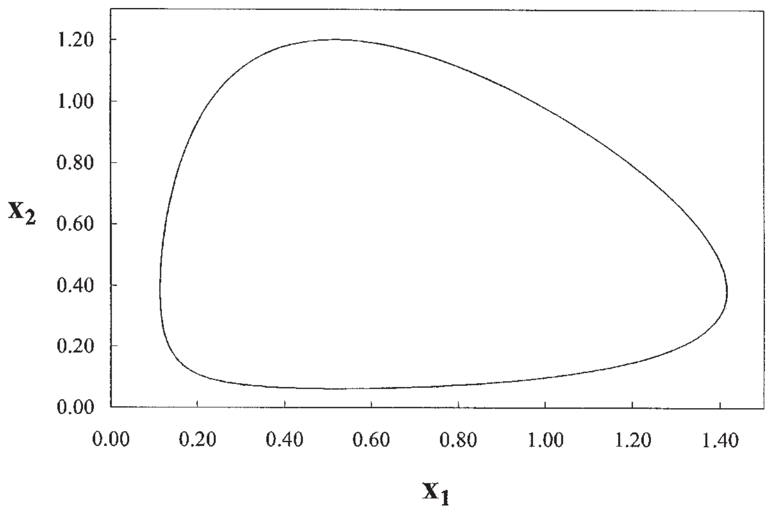

Figure 2. Phase-plane plot of equations (10) of the text for the trajectory specified in figure 1.

which reduces equations (1) to

$$\begin{aligned} \dot{x}_1 &= a x_1 + \frac{k^2}{b} x_1^{1-\alpha} \mathrm{e}^{b(x_1+x_2)/a}, \\ \dot{x}_2 &= -a\alpha x_2 - \frac{k^2}{b} x_1^{1-\alpha} \mathrm{e}^{b(x_1+x_2)/a}. \end{aligned} \tag{12}$$

Substituting equations (10) into equations (12) gives

$$\ddot{w} = a^2 \alpha w + a(1-\alpha)\dot{w} + k^2 \left[\frac{1}{b}(a\alpha w + \dot{w})\right]^{1-\alpha} \mathrm{e}^{(\alpha+1)w}. \tag{13}$$

Folding equation (13) into equation (9) (with $c = \alpha a$) yields the following equation for the first integral:

$$\dot{w}^2 + a(\alpha-1)w\dot{w} - \alpha a^2 w^2 - k^2 \left[\frac{1}{b}(\alpha a w + \dot{w})\right]^{1-\alpha} \mathrm{e}^{(\alpha+1)w} = 0. \tag{14}$$

For the case $c = a$ ($\alpha = 1$), equation (14) reduces to

$$\dot{w}^2 - a^2 w^2 - k^2 \mathrm{e}^{2w} = 0, \tag{15}$$

or

$$\dot{w} = \pm\left[a^2 w^2 + k^2 \mathrm{e}^{2w}\right]^{1/2}. \tag{16}$$

Formal integration of equation (16) leads to the quadrature

$$t - t_0 = \pm \int_w \left[a^2 w'^2 + k^2 \mathrm{e}^{2w'}\right]^{-1/2} \mathrm{d}w'. \tag{17}$$

Equation (17) represents an analytic solution to the $c = a$ LV problem. Moreover, an analysis of the form of this integral [2] shows its relationship to the family of elliptic integrals, and leads to a new class of LV related differential equations.

In order to provide a similar quadrature for the $c \neq a$ case, an alternative form of the first integral may be found by writing equations (11) as

$$-\frac{k^2}{b} x_1^{-\alpha} \mathrm{e}^{b(x_1+x_2)/a} = b x_2, \tag{18}$$

or, using equations (10),

$$-\frac{k^2}{b}\left[\frac{1}{b}(\alpha a w + \dot{w})\right]^{-\alpha} \mathrm{e}^{(\alpha+1)w} = a w - \dot{w}. \tag{19}$$

If we now define a function $\rho(w)$ such that

$$\alpha a w + \dot{w} = \alpha a \mathrm{e}^{\rho}, \tag{20}$$

and write $aw - \dot{w} = a(\alpha+1)w - (\alpha a w + \dot{w})$, equation (19) becomes

$$-\frac{k^2}{b}\left(\frac{\alpha a}{b}\right)^{-\alpha} \mathrm{e}^{(\alpha+1)w} \mathrm{e}^{-\alpha\rho} = a(\alpha+1)w - \alpha a \mathrm{e}^{\rho}, \tag{21}$$

or

$$b a(\alpha+1)w - b\alpha a \mathrm{e}^{\rho} + k^2 \left(\frac{\alpha a}{b}\right)^{-\alpha} \mathrm{e}^{(\alpha+1)w} \mathrm{e}^{-\alpha\rho} = 0. \tag{22}$$

Since $\rho(w)$ can be determined from equation (22), we have from equation (20) the first integral

$$\dot{w} = \alpha a\big(\mathrm{e}^{\rho} - w\big). \tag{23}$$

Formal integration of equation (23) yields the quadrature

$$t - t_0 = \int_w \big[\alpha a\big(\mathrm{e}^{\rho} - w'\big)\big]^{-1}\,\mathrm{d}w'. \tag{24}$$

Equation (24) represents an analytic solution to the general LV problem. As in the $c = a$ case, an analysis of the form of equation (24) [2] shows the relationship of this integral to the family of elliptic integrals, and leads to a new class of LV related differential equations.

Finally, it is worth noting that we have integrated equation (13) directly using a symbolic processor [9], and that this integration results in the same formal solution presented above (i.e., equation (24), with ρ being given by equation (22)). Using the same symbolic processor [9], we have also shown that equation (24) reduces to equation (17) under the assumption $\alpha = 1$ ($c = a$), which demonstrates the consistency of these solutions.

In summary, we have presented a transformation for the LV dynamical system that reduces this system to a single second-order autonomous ordinary differential equation. We have also provided formal analytic solutions to the LV problem by means of integral quadratures. An analytic investigation of these integrals will be presented separately [2].

Acknowledgements

This work was supported by the NLU Development Grants Program and by the Louisiana Board of Regents Support Fund. We thank an anonymous reviewer for suggestions simplifying the derivation of equation (24).

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965) pp. 896–897.
[2] C.M. Evans and G.L. Findley, Analytic solutions to a family of Lotka–Volterra related differential equations, J. Math. Chem., submitted.
[3] E. Hofelich Abate and F. Hofelich, Solution by means of Lie-series to the laser rate equations for a giant pulse including spontaneous emission, Z. Phys. 209 (1968) 13–32.
[4] E. Hofelich Abate and F. Hofelich, An analytic solution of the Lotka–Volterra equations, Z. Natursforschg. 24B (1969) 132.
[5] E.H. Kerner, Dynamical aspects of kinetics, Bull. Math. Biophys. 26 (1964) 333–349.
[6] E.H. Kerner, Note on Hamiltonian format of Lotka–Volterra dynamics, Phys. Lett. A 151 (1990) 401–402.
[7] E.H. Kerner, Comment on Hamiltonian structures for the n-dimensional Lotka–Volterra equations, J. Math. Phys. 38 (1997) 1218–1223.

[8] A.J. Lotka, Undamped oscillations derived from the law of mass action, J. Am. Chem. Soc. 42 (1920) 1595–1599.
[9] *Maple V*, Rel. 4.00a (Waterloo Maple, Inc., Waterloo, Ontario, Canada).
[10] N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, Princeton, 1962).
[11] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977), and references therein.
[12] Y. Nutku, Hamiltonian structure of the Lotka–Volterra equations, Phys. Lett. A 145 (1990) 27–28.
[13] M. Plank, Hamiltonian structures for the n-dimensional Lotka–Volterra equations, J. Math. Phys. 36 (1995) 3520–3534, and references therein.
[14] M.R. Roussel, An analytic center manifold for a simple epidemiological model, SIAM Rev. 39 (1997) 106–109.
[15] K.-i. Tainaka, Stationary pattern of vortices or strings in biological systems: Lattice version of the Lotka–Volterra model, Phys. Rev. Lett. 63 (1989) 2688–2691, and references therein.
[16] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems* (Springer, Berlin, 1990).
[17] V. Volterra, Variations and fluctuations of the number of individuals in animal species living together, in: *Animal Ecology*, ed. R.N. Chapman (McGraw-Hill, New York, 1926) pp. 409–448.